OCR
Oxford Cambridge and RSA

Monday 3 June 2019 – Morning

A Level Further Mathematics A

Y540/01 Pure Core 1

Time allowed: 1 hour 30 minutes

Model Solutions

You must have:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **16** pages. The Question Paper consists of **8** pages.

Answer **all** the questions.

1 **In this question you must show detailed reasoning.**

The quadratic equation $x^2 - 2x + 5 = 0$ has roots α and β.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. **[1]**

(b) Hence find a quadratic equation with roots $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$. **[3]**

a) $\alpha + \beta = -\frac{b}{a} = -\frac{-2}{1} = 2$

$\alpha\beta = \frac{c}{a} = \frac{5}{1} = 5$

b) $\left(\alpha + \frac{1}{\beta}\right) + \left(\beta + \frac{1}{\alpha}\right) = (\alpha + \beta) + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$

$= (\alpha + \beta) + \left(\frac{\alpha + \beta}{\alpha\beta}\right)$

$= 2 + \frac{2}{5} = \frac{12}{5} = -\frac{b}{a}$

$\left(\alpha + \frac{1}{\beta}\right) \times \left(\beta + \frac{1}{\alpha}\right) = \alpha\beta + 2 + \frac{1}{\alpha\beta}$

$= 5 + 2 + \frac{1}{5}$

$= \frac{36}{5} = \frac{c}{a}$

$x^2 - \frac{12}{5}x + \frac{36}{5} = 0$ $\times 5$

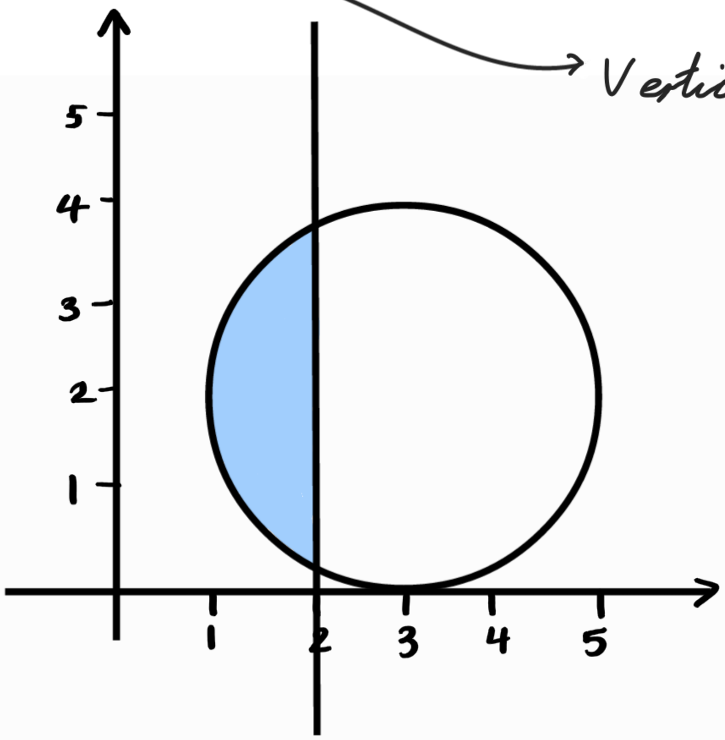
$\therefore 5x^2 - 12x + 36 = 0$

2 Indicate by shading on an Argand diagram the region

$\{z : |z| \leqslant |z-4|\} \cap \{z : |z-3-2i| \leqslant 2\}$. **[3]**

Circle, centre = $3+2i$
radius = 2

Vertical line through $2+0i$



3 **In this question you must show detailed reasoning.**

You are given that $x = 2+5i$ is a root of the equation $x^3 - 2x^2 + 21x + 58 = 0$.

Solve the equation. **[4]**

If $2+5i$ is a root, $2-5i$ is also a root (conjugate pair).

$(x-(2+5i))(x-(2-5i))$

$= x^2 - 2x + 5xi - 2x + 4 - 10i - 5xi + 10i - 25i^2$

$= x^2 - 4x + 29$

• $(ax+b)(x^2-4x+29)$

compared coefficients:

x^3 coeff: $a = 1$

constant: $29b = 58$

$b = 2$

$\therefore (x+2)(x^2-4x+29)$

So the solution of the cubic is:

$x = -2, 2 \pm 5i$

4 Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that $\sum_{r=1}^{10} r(3r-2) = 1045$. **[3]**

$$\sum_{r=1}^{10} r(3r-2) = \sum_{r=1}^{10} 3r^2 - 2r$$

$$= 3\sum_{r=1}^{10} r^2 - 2\sum_{r=1}^{10} r$$

$$= 3\left(\frac{n}{6}(n+1)(2n+1)\right) - 2\left(\frac{n}{2}(n+1)\right)$$

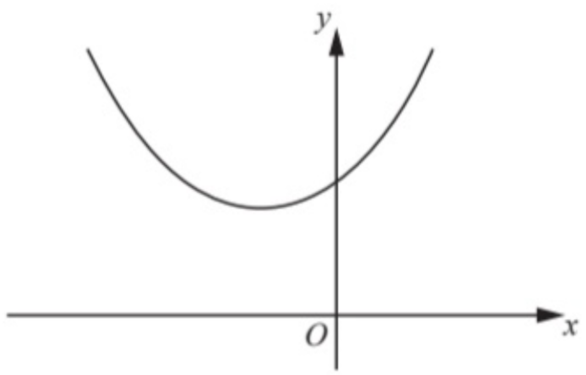
when $n=10$

$$3\left(\frac{10}{6} \times 11 \times 21\right) - 2\left(\frac{10}{2}(11)\right)$$

$$= 1155 - 110$$

$= 1045$ (as required)

5 The diagram shows part of the curve $y = 5\cosh x + 3\sinh x$.



(a) Solve the equation $5\cosh x + 3\sinh x = 4$ giving your solution in exact form. **[4]**

(b) In this question you must show detailed reasoning.

Find $\int_{-1}^{1}(5\cosh x + 3\sinh x)\,dx$ giving your answer in the form $ae + \frac{b}{e}$ where a and b are integers to be determined. **[3]**

a)

$$5\left(\frac{e^x + e^{-x}}{2}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right) = 4$$

$$5e^x + 5e^{-x} + 3e^x - 3e^{-x} = 8$$

$\div 2$ $\quad 8e^x + 2e^{-x} = 8 \quad$ $\div 2$

$\times e^x \quad 4e^x + e^{-x} = 4 \quad \times e^x$

$$4e^{2x} + 1 = 4e^x$$

$$4e^{2x} - 4e^x + 1 = 0$$

$$(2e^x - 1)^2 = 0$$

$$e^x = \frac{1}{2}$$

$$\therefore x = -\ln 2$$

b) $\int_{-1}^{1} 5\cosh x + 3\sinh x \, dx$

$= \left[5\sinh x + 3\cosh x\right]_{-1}^{1}$

$= \left[5\left(\frac{e^1 - e^{-1}}{2}\right) + 3\left(\frac{e^1 + e^{-1}}{2}\right)\right] - \left[5\left(\frac{e^{-1} - e^{1}}{2}\right) + 3\left(\frac{e^{-1} + e^{1}}{2}\right)\right]$

$= (4e^1 - 4e^{-1}) - (4e^{-1} - e^1)$

$= 5e - 5e^{-1}$

$= 5e - \frac{5}{e}$

6 You are given that $y = \tan^{-1}\sqrt{2x}$.

(a) Find $\frac{dy}{dx}$. **[2]**

(b) Show that $\int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\sqrt{x}}{(x+2x^2)} dx = k\pi$ where k is a number to be determined in exact form. **[4]**

a) $\tan y = \sqrt{2} \times x^{\frac{1}{2}}$

$\sec^2 y \times \frac{dy}{dx} = \frac{\sqrt{2}}{2} x^{-\frac{1}{2}}$

As $\sec^2 y = 1 + \tan^2 y$

$(1 + \tan^2 y)\frac{dy}{dx} = \frac{\sqrt{2}}{2} x^{-\frac{1}{2}}$

As $\tan y = \sqrt{2}\, x^{\frac{1}{2}}$
then $1 + \tan^2 y = 1 + 2x$

$(1+2x)\frac{dy}{dx} = \frac{\sqrt{2}}{2} x^{-\frac{1}{2}}$

$\therefore \frac{dy}{dx} = \frac{\sqrt{2}}{2} x^{-\frac{1}{2}} \times \frac{1}{1+2x}$ or $= \frac{\sqrt{2}}{2\sqrt{x}} \times \frac{1}{1+2x}$

b) let $u = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$dx = 2x^{\frac{1}{2}}\,du$

As $u = x^{\frac{1}{2}}$

$dx = 2u\,du.$

$$\int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\sqrt{x}}{x+2x^2}\,dx = \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{u}{u^2+2u^4} \times 2u\,du = 2\int_{\frac{1}{6}}^{\frac{1}{2}} \frac{1}{1+2u^2}\,du$$

$$= \sqrt{2}\left[\tan^{-1} u\sqrt{2}\right]_{\frac{1}{6}}^{\frac{1}{2}}$$

$$= \sqrt{2}\left[\tan^{-1}\sqrt{2x}\right]_{\frac{1}{6}}^{\frac{1}{2}} = \sqrt{2}\left(\tan^{-1}1 - \tan^{-1}\frac{1}{\sqrt{3}}\right) = \sqrt{2}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{\pi\sqrt{2}}{12}$$

7 The function $\operatorname{sech} x$ is defined by $\operatorname{sech} x = \dfrac{1}{\cosh x}$.

(a) Show that $\operatorname{sech} x = \dfrac{2e^x}{e^{2x}+1}$. **[2]**

(b) Using a suitable substitution, find $\int \operatorname{sech} x \, dx$. **[4]**

a) $\cosh x = \dfrac{e^x + e^{-x}}{2} \times \dfrac{e^x}{e^x} = \dfrac{e^{2x}+1}{2e^x}$

$\therefore \dfrac{1}{\cosh x} = \operatorname{sech} x = \dfrac{2e^x}{e^{2x}+1}$ (as required)

b) let $u = e^x$

$du = e^x \, dx$

$dx = \frac{du}{u}$

$$\int \operatorname{sech} x \, dx = \int \frac{2e^x}{e^{2x}+1} \, dx = \int \frac{2\cancel{u}}{u^2+1} \times \frac{du}{\cancel{u}}$$

$$= 2\tan^{-1}(u) + c$$

$$= 2\tan^{-1}(e^x) + c$$

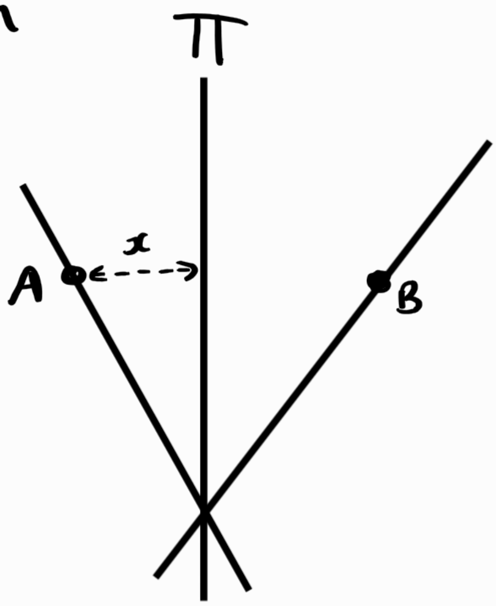
8 The equation of a plane is $4x + 2y + z = 7$.
The point A has coordinates (9, 6, 1) and the point B is the reflection of A in the plane.

Find the coordinates of the point B. **[6]**

AB has a direction vector of $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

$$A: \underline{r} = \begin{pmatrix} 9 \\ 6 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

If $\begin{pmatrix} 9+4\lambda \\ 6+2\lambda \\ 1+\lambda \end{pmatrix}$ lies on the plane then



$$4(9+4\lambda) + 2(6+2\lambda) + (1+\lambda) = 7$$

$$36 + 16\lambda + 12 + 4\lambda + 1 + \lambda = 7$$

$$49 + 21\lambda = 7$$

$$21\lambda = -42$$

$$\therefore \lambda = -2$$

So B is where $\lambda = 2 \times -2 = -4$

when $\lambda = -4$

B coordinates: $(9-16, 6-8, 1-4) \Rightarrow (-7, -2, -3)$

9 In this question you must show detailed reasoning.

You are given the complex number $\omega = \cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi$ and the equation $z^5 = 1$.

(a) Show that ω is a root of the equation. **[2]**

(b) Write down the other four roots of the equation. **[1]**

(c) Show that $\omega + \omega^2 + \omega^3 + \omega^4 = -1$. **[2]**

(d) Hence show that $\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1 = 0$. **[3]**

(e) Hence determine the value of $\cos\frac{2}{5}\pi$ in the form $a + b\sqrt{c}$ where a, b and c are rational numbers to be found. **[4]**

a) $w = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$$w^5 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5$$

$$= \cos 2\pi + i\sin 2\pi = 1$$

b) roots: $w^2, w^3, w^4, 1$

c) $z^5 = 1$

$\Rightarrow w^5 - 1 = 0$

$\therefore (w-1)(w^4 + w^3 + w^2 + w + 1) = 0$

As $w^4 + w^3 + w^2 + w + 1 = 0$

$\Rightarrow w^4 + w^3 + w^2 + w = -1$ (as required)

d) $\left(\omega+\frac{1}{\omega}\right)^2+\left(\omega+\frac{1}{\omega}\right)-1$

$= \omega^2+2+\frac{1}{\omega^2}+\omega+\frac{1}{\omega}-1$

$= \omega^2+\omega+1+\frac{1}{\omega}+\frac{1}{\omega^2}$

$= \frac{1}{\omega^2}\left(\omega^4+\omega^3+\omega^2+\omega+1\right) = 0$ (as required)

Since $\frac{1}{\omega^2} \neq 0$, $\omega^4+\omega^3+\omega^2+\omega+1=0$

e) $\frac{1}{\omega} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

$\left(\omega+\frac{1}{\omega}\right) = 2\cos\frac{2\pi}{5}$

As $\left(\omega+\frac{1}{\omega}\right) = \frac{-1\pm\sqrt{5}}{2}$

then $2\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{2}$ $\therefore$ $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$

$= -\frac{1}{4}+\frac{\sqrt{5}}{4}$

10 You are given the matrix $\mathbf{A}$ where $\mathbf{A} = \begin{pmatrix} a & 2 & 0 \\ 0 & a & 2 \\ 4 & 5 & 1 \end{pmatrix}$.

(a) Find, in terms of a, the determinant of $\mathbf{A}$, simplifying your answer. **[2]**

(b) Hence find the values of a for which $\mathbf{A}$ is singular. **[2]**

You are given the following equations which are to be solved simultaneously.

$$\begin{aligned} ax + 2y \quad &= 6 \\ ay + 2z &= 8 \\ 4x + 5y + z &= 16 \end{aligned}$$

(c) For each of the values of a found in part **(b)** determine whether the equations have

- a unique solution, which should be found, or
- an infinite set of solutions or
- no solution. **[7]**

a) $\det A = a(a-10) - 2(-8) + 0(0-4a)$

$= a^2 - 10a + 16$

b) If A is singular, then $\det A = 0$

$a^2 - 10a + 16 = 0$

$(a-2)(a-8) = 0$

$\therefore a = 2$ & $a = 8$

c) when $a = 2$

- $2x + 2y = 6$ — 1
- $2y + 2z = 8$ — 2
- $4x + 5y + z = 16$ — 3

As $2 \times 1 + \frac{1}{2} 2 = 3$

$4x + 4y + y + z = 12 + 4$

$4x + 5y + z = 16$ $\therefore$ an infinite set of solutions.

when $a=8$

- $8x+2y=6$ — ①
- $8y+2z=8$ — ②
- $4x+5y+z=16$ — ③

As $\frac{1}{2}\times$ ① $+\frac{1}{2}$ ② $\neq$ ③

$4x+y+4y+z=3+4$

$4x+5y+z=7\neq 16$

$\therefore$ no solution

11 A particle is suspended in a resistive medium from one end of a light spring. The other end of the spring is attached to a point which is made to oscillate in a vertical line.

The displacement of the particle may be modelled by the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=10\sin t$$

where x is the displacement of the particle below the equilibrium position at time t.

When $t=0$ the particle is stationary and its displacement is 2.

(a) Find the particular solution of the differential equation. **[11]**

(b) Write down an approximate equation for the displacement when t is large. **[2]**

END OF QUESTION PAPER

a) $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=10\sin t$

$m^2+2m+5=0$

$\therefore m=-1\pm 2i$

$\Rightarrow x=e^{-t}(A\cos 2t+B\sin 2t)$

P.I : $x = \lambda \sin t + \mu \cos t$

$\frac{dx}{dt} = \lambda \cos t - \mu \sin t$

$\frac{d^2x}{dt^2} = -\lambda \sin t - \mu \cos t$

$\Rightarrow (-\lambda \sin t - \mu \cos t) + 2(\lambda \cos t - \mu \sin t) + 5(\lambda \sin t + \mu \cos t)$

$= 10 \sin t$

comparing $\sin t$ coefficients:

$-\lambda - 2\mu + 5\lambda = 10$

$4\lambda - 2\mu = 10$

comparing $\cos t$ coefficients:

$-\mu + 2\lambda + 5\mu = 0$

$4\mu = -2\lambda$

$\therefore -(2 \times 4\mu) - 2\mu = 10$

$-10\mu = 10$

$\mu = -1$

$\therefore \lambda = 2$

G.S : $x = e^{-t}(A \cos 2t + B \sin 2t) + 2 \sin t - \cos t$

when $t=0$, $x=2$

$2 = A - 1$

$A = 3$

when $t=0$, $\frac{dx}{dt}=0$

$$\frac{dx}{dt} = -e^{-t}(A\cos 2t + B\sin 2t) + e^{-t}(-2A\sin 2t + 2B\cos 2t)$$
$$+ 2\cos 2t + \sin t$$

$0 = -A + 2B + 2$

As $A = 3$

$0 = -3 + 2B + 2$

$1 = 2B$

$\therefore B = \frac{1}{2}$

$$\Rightarrow x = e^{-t}\left(3\cos 2t + \frac{1}{2}\sin 2t\right) + 2\sin t - \cos t$$

b) $x = e^{-t}\left(3\cos 2t + \frac{1}{2}\sin 2t\right) + 2\sin t - \cos t$

As $t \to \infty$, $e^{-t} \to 0$

$\therefore x \approx 2\sin t - \cos t$.